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RF-3491

M. Sc. (Part - I) Examination

April / May – 2010

Mathematics : Paper - 405

(Ordinary Differential Equations)

(New Course)

Time : Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ M. Sc. (Part - 1)

Name of the Subject :
☛ Mathematics-405 (New)

☛ Subject Code No. : 3 4 9 1 ☛ Section No. (1, 2,.....) : Nil

Seat No. :

Student's Signature

(2) Attempt all questions

(3) Figure to the right indicates marks.

(4) Notations and conventions are all standard.

1 (a) Let $g(t,u)$ be defined on the rectangle **5**

$R(a,b): t_0 \le t \le t_0 + a,\ |u-u_0| \le b$ and is measurable in t for each fixed a and continuous in u for each fixed t. If there exists a Lebesgue integrable function $m(t)$ on $[t_0, t_0+a]$ such that $|g(t,u)| \le m(t)$ for $(t,u) \in R(a,b)$ then prove that initial value problem $u' = g(t,u)$, $u(t_0) = u_0$ has at least one solution $u(t)$ on some interval $[t_0, t_0+\beta]$, $B > 0$.

(b) Let us assume that $f(t,x)$ is continuous on $B_0 : t_0 \le t \le t_0 + a,\ \|x - x_0\| \le b$, where a and b are positive real numbers and satisfy the Lipschitez condition in B_0. Let $M = \max_{(t,x)\in B_0} \|f(t,x)\|$, $\propto = \min\left(a, \frac{b}{M}\right)$ then prove that the initial value problem $x' = f(t,x), x(t_0) = x_0$ has a unique solution $x(t)$ on $[t_0, t_0 + a]$.

(c) Apply Picard's method to solve the initial value problem $u' = 4t + 2tu, 4(0) = 1$. **4**

OR

1 (a) Let the function $g(t,u)$ be continuous and bounded in the strip $S : t_0 \le t \le t + a, |u| < \infty, -$ then prove that the initial value problem $u' = g(t,u); u(b) = u_0$ has at least one solution $u(t)$ defined on the interval $[t_0, t_0 + a]$. **5**

(b) If $g(t,u)$ is a continuous function of t and u in a closed bounded region $R(a,b)$ satisfies the Lipschitz condition in R then prove that there exists a unique solution $u(t)$ to the initial value problem $u' = g(t,u), u(t_0) = u_0$ on the interval $J : |t - t_0| \le h$, where $h = \min\left\{a, \frac{b}{m}\right\}$ and $|g(t,u)| \le M$ for $(t,u) \in D$ and $D : |u - u_0| \le h, |u - u_0| \le b$. **5**

(c) Show that $g(t,u)=t^2u+u^2$ satisfies the Lispschitz condition on $R:|t|\le 2 \quad |u-1|\le 2$. 4

2 (a) Prove that a complex number λ is a characteristics exponent of $x'=A(t)x$, where $A(t,+w)=A(t),(w\neq 0)$ if there exists a non-trivial solution of $x'=A(t)x$ in the form $e^{\lambda t}p(t)$, where $p(t+w)=p(t)$. 5

(b) If all the solutions of $x'=Ax$, where A is $n\times n$ constant matrix and x is an n-vector are bounded on $[0,\infty)$, then prove that all the solutions of $x'=(A+B(t))x$ are also bounded in $[0,\infty)$, provided $\int^{\infty}\|B(S)\|\,ds<\infty$. 5

(c) Define stable characteristic polynomial. Determine the stable characteristic polynomial for the equation $y'''+2y''+3y'+4y=0$. 4

OR

2 (a) Prove that a system $x'(t)=A(t)x(t)$ is uniformly stable if it is stable and reducible. 5

(b) Define adjoint system to system $x'=A(t)x(t)$. 5

Prove that if Φ is a fundamental matrix of $x'=A(t)x$ then ψ is a fundamental matrix of its adjoint system if and only if $\psi^T\Phi=C$, where C is a constant non-singular matrix.

(c) Discuss the solution of Mathieu's equation 4

$u'' + \left(\delta^2 + \in \cos 2t\right)u = 0$ with initial conditions

$u_1(0) = 1,\ u_1'(0) = 0, u_2(0) = 0, u_2'(0) = 1$.

3 (a) Let $\Phi(t)$ be a fundamental matrix of $x' = A(t)x$ with 5

$\Phi(t_0) = I$. Then prove that $x' = A(t)x$ is strongly stable if and only if there exists a positive constant M such that $\|\Phi(t)\| \le M, \|\Phi^{-1}(t)\| \le M$ for $t \ge t_0$.

(b) Prove that all the solutions of $x' = A(t)x$ where 5

$A(t)$ is an $n \times n$ continuoues matrix on $[0, \infty)$ and x is an n-vector, are stable if and only if they are bounded.

(c) Show that the equation $u'' + \frac{2u'}{t+1} = 0$ is stable but 4

not uniformly stable.

OR

3 (a) Let $\Phi(t)$ be a fundamental matrix of $x'(t) = A(t)x(t)$ 5

with $\Phi(t_0) = I$. Then prove that $x'(t) = A(t)x(t)$ is uniformly stable if and only if there exists a positive constant M such that $\|\Phi(t)\Phi^{-1}(s)\| \le M$ for $t_0 \le s < t < \infty$.

(b) Prove that if all the characteristic roots of A have negative real parts, then every solution of $x' = Ax$, where $A = (a_{ij})$ is constant matrix is asymtotically stable. **5**

(c) Determine the type of stability of the critical point $(0, 0)$ of each of the following linear systems and sketch the phase potraits **4**

(1) $x_1' = 3x_1 + 2x_2$
$x_2' = 4x_1 - x_2$

(2) $x_1' = x_1 + 3x_2$
$x_2' = -6x_1 + 5x_2$

4 (a) If all the solutions of $u'' + a(t)u = 0$ are bounded on $[0, \infty)$ then prove that all the solutions of $u'' + (a(t) + b(t))u = 0$ are also bounded on $[0, \infty)$, provided $\int_0^\infty |b(t)\,dt| < \infty$. **5**

(b) Let $a(t)$ be a continuoues differentiable function for $t\,E\,[0, \infty)$ then all the solutions of $u'' + a(t)u = 0$ are bounded on $[0, \infty)$ provided that $a(t) \to \infty$ monotonically as $t \to \infty$. **5**

(c) For the equation $y''+(1+b(t))y=0$ show that

$y(t)=\exp\left[\int_1^t \frac{\sin^2 s}{s} ds\right]\sin t$ is the solution and it

is not bounded, where $b(t)=\frac{2\sin 2t}{t}\left[\frac{\sin 2t}{8t}-1\right]; t\geq 0$.

OR

4 (a) If $\|u\|$ and $\|u''\|$ are bounded then prove that $\|u'\|$ is also bounded. 5

(b) Let $b(t)$ be the continuously differentiable on $[0,\infty)$. If $b(t)\to 0$ as $t\to\infty$ and $\int_0^\infty |b'(s)|ds<\infty$ then show that all the solutions of $u''+(1+b(t))u=0$ are bounded over $[0,\infty)$ 5

(c) Let $u_1(t)$ and $u_2(t)$ be two linearly independent solutions of $V''+\alpha(t)V=0$ on the interval $0\leq +\infty$. Then show that the general solution $u(t)$ of the inhomogeneous equation $u''+\alpha(t)u=g(t)$ is given by 4

$u(t)=C_1u_1(t)+C_2u_2(t)$

$+\int_0^t \left[u_1(s)u_2(t)-u_1(t)u_2(s)\right]g(s)ds$, where C_1 and C_2 are constants.

5 Attempt any **two** : **14**

(1) Show that for vander pol equation

$$u'' + \in \left(u^2 - 1\right)u' + u = 0$$, where $\in$ is a positive constant;

the critical point $(0,0)$ is always unstable.

(2) For the differential equation

$$u' = \begin{cases} u^2 \sin^2\left(\frac{1}{4}\right)u & ; u \neq 0 \\ 0 & ; u \neq 0 \end{cases}$$

show that the zero solution is uniformly stable but not asymptotically stable.

(3) Solve the first order inhomogeneous system

$$x_1' = 3x_1 - x_2 + 1$$

$$x_2' = 4x_1 - x_2 + t$$